Supplementary Data:
Implications of Liebig’s Law of the Minimum for tree-ring reconstructions

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S1 Percentile correlations for individual sites.

The signature of the law of the minimum – as diagnosed in figure 2 of the main manuscript by comparing temperature correlations across percentiles – is also clearly visible at the site level at almost all sites. Here we replicate figure 2(c) for all 207 sites used in the study (figure S1: pages 1 to 5). The sites are ordered, from highest to lowest, by the correlation of the mean chronology with local temperature. Because we show only one site per figure panel, there is no need to normalize the correlation statistics, and we report the raw $R^2$ values recovered from correlating each percentile with local temperature at each site on the y-axis of figure S1 (red lines). For comparison, we also indicate the correlation of the mean chronology at each site with local temperature (figure S1, black lines). For each site, we mark the percentile with the highest correlation with temperature with a black dot. The two sites where the mean correlation with temperature is higher than the correlation of any percentile with temperature are instead indicated by coloring the entire percentile correlation line black instead of red. The y-axis scale is consistent within each of the 5 pages, but this scale is different between the pages. Site name, latitude, and longitude are reported as text inside each figure caption.
Figure S1  Page 1 of 5 Comparison of local temperature signal at each percentile at each site to temperature signal in mean chronology. See “S1 Percentile correlations for individual sites” section text for details.
Supplementary Figure S1: Page 2 (of 5); Panels 50 to 100 (of 207)

Figure S1  Page 2 of 5 Continued from previous page.
Supplementary Figure S1: Page 3 (of 5); Panels 100 to 150 (of 207)

Figure S1 Page 3 of 5 Continued from previous page.
Supplementary Figure S1: Page 4 (of 5); Panels 150 to 200 (of 207)

Figure S1  Page 4 of 5 Continued from previous page.
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<tr>
<th>Location</th>
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<th>Longitude</th>
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</tbody>
</table>

Supplementary Figure S1: Page 5 of 5; Panels 200 to 207 (of 207)

$R^2$ percentile

Figure S1 Page 5 of 5 Continued from previous page.
Model $\Delta$coherence in response to Red, Blue and White Noise.

The difference between the coherence of percentile chronologies and the mean chronology ($\Delta$coherence) provides another means of examining the operation of Liebig’s Law in tree-ring density records. The sense of the result is that if tree-level noise is controlled by Liebig’s Law ($H_1$ and $H_{outlier}$) then coherence between temperature and tree-ring percentiles should be larger than the coherence between temperature and the mean tree-ring chronology ($\Delta$coherence $>0$) at high percentiles. Higher coherence for high percentile chronologies should be especially the case at frequencies where the forcing timeseries (the CGF) is most energetic. In contrast, if the noisy thermometer model of trees ($H_0$) holds, then coherence between temperature and tree-ring percentiles is expected to be smaller than the coherence between temperature and the mean tree-ring chronology ($\Delta$coherence $\leq 0$) at all percentiles, and this difference in coherence should be especially large at frequencies where the forcing timeseries (the CGF) is most energetic. Thus $\Delta$coherence for high percentiles at the most energetic frequencies provides a particularly powerful test for operation of Liebig’s Law at the tree level because it is where the both null and the alternative models predict strong but opposing responses.

To illustrate this difference between the predictions of the models, and to illustrate that differences in $\Delta$coherence across frequency space are controlled by the spectrum of the CGF, we model the response of $H_0$, $H_1$ and $H_{outlier}$ to red, white and blue noise CGF forcing. We model each of the 207 sites used in this study using the fixed parameter values described in the main text and in the parameter estimation sections of this supplement (sections S3 and S4). We calculate $\Delta$coherence for each percentile at each station as the difference between the coherence between local temperature and the percentile time series and the coherence between local temperature and the mean site chronology. Positive values indicate that the temperature signal is stronger in the percentile time series. For each station we average over 10 random realizations of the model to give stabilized estimates of the expected behavior.

Results of this analysis are shown in figure S2.
Figure S2  Difference in coherence with temperature between percentile chronologies and the mean chronology in response to spectrum of CGF forcing. \(\Delta\)coherence for each of the three models used in this manuscript: \(H_0\) (column 1), \(H_1\) (column 2), \(H_{1\text{out}}\) (column 3). Row 1 is driven by red noise. Row 2 is driven by white noise. Row 3 is driven by blue noise. Further details given in supplemental text under heading “S2 Model \(\Delta\)coherence in response to Red, Blue and White Noise.”
S3 Parameter Estimation for the Null Model (H₀).

The Null Model (Eq. 1) contains 2 free parameters. These are:

1. P₁ the signal-to-noise ratio of the CGF, where temperature is treated as signal, and all other variability is treated as noise.
2. P₂ the variance of the LGF distribution.

In this model the CGF is always normalized to unit variance and so does not constitute an independent parameter.

In practice, the sense of the model result as represented in figure 2(a) is inherent to the functional form of H₀ (Eq. 1), and not on the particular parameters chosen. However, for purposes of comparison, we seek parameters that have a consistency with the variability in the tree ring observations.

We tune the two free parameters in H₀ to match:

- M₁ the correlation at each site between the mean tree-ring chronology and local temperature.
- M₂ the average correlation at each site between the mean tree-ring chronology and each individual standardized tree-ring series.

The model parameter P₁ and P₂ respectively exert the greatest control on the M₁ and M₂ tuning metrics.

Formally, we pick P₁ and P₂ to minimize the cost function J₇₀:

\[ J_{H₀} = \frac{1}{207} \sum_{n=1}^{207} (M₁^{\text{model}}(n) - M₁^{\text{obs}}(n))^2 + (M₂^{\text{model}}(n) - M₂^{\text{obs}}(n))^2 \]  

(S1)

where n indicates the site number for each of the 207 sites used in the analysis, the “obs” superscript indicates that metric is calculated using the observed tree-ring density values for the site and the “model” superscript indicates that the metric is calculated using the values simulated using
Eq. 1. We use a brute-force search technique to minimize $J_{H0}$ in which we map out the value of the cost function for all values of $P_1$ from 0.1 to 2.5 and for all values of $P_2$ from 0 to 3, each at steps of 0.1 (figure S3).

We choose the model parameters $P_1$ and $P_2$ that give the minimum values for $J_{H0}$. These are $P_1 = 0.6$ and $P_2 = 1.4$. A map of the cost function, $J_{H0}$, as a function $P_1$ and $P_2$ is shown in figure S3.

**Figure S3** Map of the natural logarithm of the cost function $J_{H0}$. X-axis shows parameter $P_1$. Y-axis shows parameter $P_2$. The magenta X indicates the minimum value for the cost function, which is found at $P_1 = 0.6$ and $P_2 = 1.4$. 
Parameter Estimation for the Alternative Model (H₁).

The Alternate Model (Eq. 2) contains 3 free parameters. These are:

- **P₁**: the signal-to-noise ratio of the CGF, where temperature is treated as signal, and all other variability is treated as noise.
- **P₂**: the variance of the LGF distribution. The CGF is always normalized to unit variance.
- **P₃**: the mean value of the LGF distribution. The CGF is always standardized to zero mean.

As with our null model, we seek parameters that have a consistency with the variability in the tree ring observations.

In this case, the three free parameters in H₁ are specified through the following measures:

- **M₁**: the correlation at each site between the mean tree-ring chronology and local temperature.
- **M₂**: the average correlation at each site between the mean tree-ring chronology and each individual standardized tree-ring series.
- **M₃**: the average variance across trees within an individual site, calculated separately for each year and then averaged across all record years.

The first model parameter, P₁, primarily controls M₁, whereas P₂ and P₃ jointly constrain M₂ and M₃.

Formally, we pick P₁, P₂ and P₃ to minimize the cost function J_{H₁}:

\[
J_{H₁} = \frac{1}{207} \sum_{n=1}^{207} (M₁^{\text{model}}(n) - M₁^{\text{obs}}(n))^2 + (M₂^{\text{model}}(n) - M₂^{\text{obs}}(n))^2 + (M₃^{\text{model}}(n) - M₃^{\text{obs}}(n))^2, 
\]

(S2)
We use a brute-force search technique to minimize $J_{H1}$ in which we map out the value of the cost function for all values of $P_1$ from 0.1 to 1.5, for all values of $P_2$ from 0 to 1.5, and for all values of $P_3$ from -1 to 1, each with a 0.1 resolution (figure S4). The model parameters that give the minimum values for $J_{H1}$ are $P_1 = 0.7$, $P_2 = 1$ and $P_3 = 0.3$. A map of the cost function ($J_{H1}$) as a function $P_1$, $P_2$ and $P_3$ is shown in figure S4.
**Figure S4** Map of the natural logarithm of the cost function $J_{H1}$. Each panel shows the minimum of the cost function across one of the three dimensions of the cost-function map.  
(a) X-axis shows parameter $P_1$. Y-axis shows parameter $P_2$. Contoured values indicate the minimum values across computed values of $P_3$ for $\log(J_{H1})$. The magenta X indicates the coordinates of the computed global minimum value for the cost function.  
(b) Same as (a), but with parameter $P_2$ on the x-axis and parameter $P_3$ on the y-axis and with contoured values indicating the minimum values of $\log(J_{H1})$ across computed values of $P_1$.  
(c) Same as (a), but with parameter $P_1$ on the x-axis and $P_3$ on the y-axis and with contoured values indicating the minimum values of $\log(J_{H1})$ across computed values of $P_2$. 
