Newton's Second Law and the Hydrostatic Relation

Quantities	Dimensions	MKS Units
mass	mass	kg
velocity or motion	distance/time	m/s
acceleration (rate of change of velocity with respect to time)	(distance/time)/time or distance/time^2	m/s^2
force ("push" or "pull" that can change motion)	mass x acceleration or mass x distance/time^2	kg x m/s^2 (Newton)
(Note that <u>weight</u> is just a special case of a force--the force of gravity)		
pressure (collective force exerted by random molecular collisions against each unit of area of a an air parcel's surroundings	force/area or (mass x distance/s^2)/area	(kg x m/s^2)/ m^2 (Pascal)

Newton's Second Law:

> **acceleration**   =   [**sum of forces**]/**mass**   =   **sum of [forces/mass]**
> of the object   acting on the object   acting on the object

We can think of the terms on the right-hand-side of this relation—namely, the forces acting on each unit of mass of an object—as the *causes* of changes in the motion of the object. The left-hand-side represents the *effect* of those causes—namely, the acceleration, defined as the rate at which the motion (i.e., the velocity) of the object changes with respect to time.

This relation applies to different directions independently. For example, the acceleration of an object in the vertical direction is caused by the sum of the forces acting on an object in the vertical direction, independent of the acceleration and sum of forces acting on the object in horizontal directions (and vice-versa).

Example

Consider a cubical parcel of air in the atmosphere. Assume that this parcel (and the air immediately around it) is not moving vertically (or horizontally) and that the forces acting on it in the vertical direction balance each other (that is, they sum to zero). Newton's Second Law implies that the rate at which the parcel changes its vertical motion—that is, its vertical acceleration—is zero.

Under our simplifying assumptions, the forces acting on this parcel include:

$F_g \equiv$ the force exerted by gravity (that is, the parcel's weight);

$F_{Pbot} \equiv$ the force exerted upward on the bottom of the parcel by pressure of the air below the parcel; and

$F_{Ptop} \equiv$ the force per unit mass exerted downward on the top of the parcel by pressure of the air above the parcel.

By convention we'll agree that upward is the positive direction. With this convention, $F_g < 0$, $F_{pbot} > 0$, and $F_{ptop} < 0$. (The signs of these vertically-acting forces are determined by the direction—up or down—in which they act.

Our assumption that these forces balance can be stated mathematically as:

$$0 = F_g + F_{p_{bot}} + F_{p_{top}}$$

(In the diagram below, the altitudes of the top and bottom of the parcel above sea level are represented by z_{top} and z_{bot}, respectively. The difference between them,

which is just the height of the parcel, we represent with the symbol Δz.)

$A \equiv$ area covered by top or bottom of the parcel
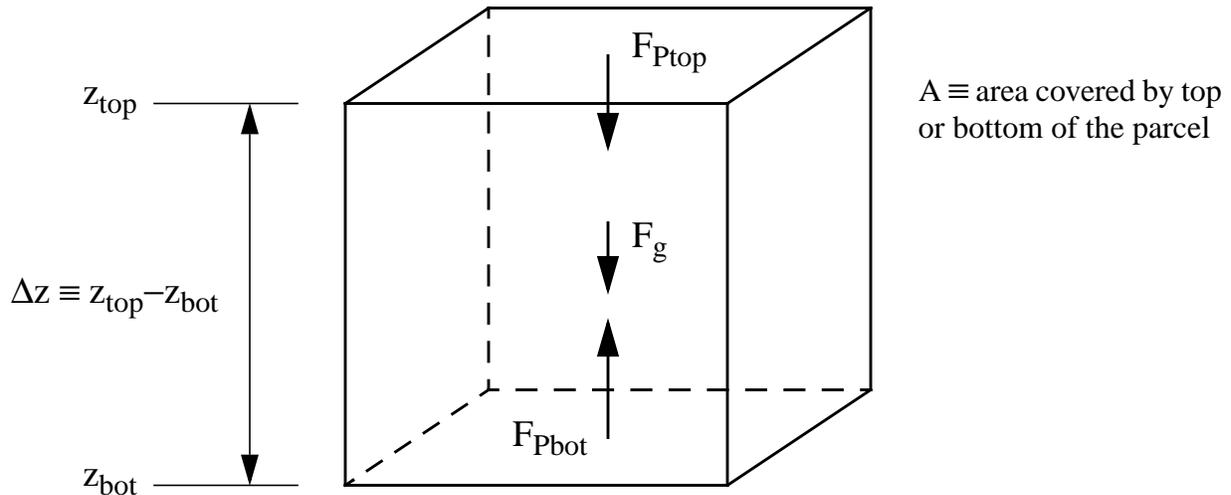
We'd prefer to relate these forces to quantities more familiar to us—they seem a little abstract in their current form. Let's start with the force of gravity.

Consider Newton's Second Law applied to an object of mass $\equiv m$, on which the force of gravity is the only force acting. In that case,

$$\text{acceleration} = F_g/m$$

The right-hand side represents the force of gravity acting on *each unit* of mass of the object (i.e., the gravitational force per unit mass).

In some famous experiments supposedly done in the 1500's, Galileo dropped objects with different masses from the tower of Pisa and observed that they hit the ground at the same time. This demonstrated that the *acceleration* produced by the gravity, acting alone on an object, doesn't depend on the object's mass.

The acceleration does vary inversely with the square of the distance between the centers of mass of two objects "attracting" each other gravitationally. However, about 99% of the air in the atmosphere lies within 50 km of the earth's surface, so the distance of most air parcels from the center of the earth, about 6,370 km away, effectively hardly varies. Hence, for most purposes we can treat the acceleration produced by gravity acting alone as essentially a constant. Taking into account our sign convention (positive upward, negative downward):

$$-g_0 \approx F_g/m$$

In MKS units, $g_0 = 9.81$ m/s^2. By convention it's a positive number, so we have to attach the minus sign to indicate that the acceleration would be downward.

It usually isn't convenient to work with the mass of a particular parcel, so we'd prefer to relate the mass to other quantities with which we're familiar. A good candidate is the density, which is defined as the mass of an object per unit of volume occupied by the object. In our case, if we represent the volume of our parcel with the symbol "V", we can define the density as:

$$\rho \equiv m/V$$

(When we divide the mass by the volume, the resulting number represents the amount of mass in *each unit* of volume.) In our case, since we've conveniently chosen a cubical parcel, we can relate the volume occupied by the parcel to its depth, Δz, and to the area covered by its top/bottom, A:

$$V = A \times \Delta z$$

and so we can relate the density of the parcel to A and Δz:

$$\rho = m/(A \times \Delta z)$$

and solving for the mass gives us:

$$m = \rho \times (A \times \Delta z)$$

Substituting for the mass in the expression for F_g on the previous page gives:

$$F_g = -(\rho \times A \times \Delta z) \times g_0$$

where the factors in parentheses represent the mass. This now relates the force of gravity acting on the parcel to other quantities that will be easier for us to work with.

Now lets look at the forces due to pressure. Since pressure is defined as the

collective force exerted by collisions of randomly-moving molecules against *each unit* of area of their surroundings, it follows that the total force exerted by pressure on the bottom of the parcel would be the pressure there (p_{bot}, force per unit area) multiplied by the number of units of area, A, covered by the bottom of the parcel:

$$F_{Pbot} = p_{bot} \text{ x } A$$

Since both pressure and area are always positive quantities, and because F_{Pbot} is directed upwards and is therefore positive (by our convention that upwards is a positive direction), we don't need to include any minus signs.

Similarly, the force exerted on the top of the parcel by pressure there (p_{top}) is:

$$F_{Ptop} = -p_{top} \text{ x } A$$

This time we have to include a minus sign, because F_{Ptop} is directed downward and is therefore negative, whereas both P_{top} and A are positive.

Now lets go back to the equation on Page 2, which we got by applying Newton's Second Law to the parcel under the assumption that the three forces acting on it balance (that is, sum to zero), and substitute the various alternative expressions we've come up with for the three forces:

$$0 = -(\rho A \Delta z)g_0 + p_{bot}A - p_{top}A$$

(where we've dropped the annoying "x" between factors, leaving the multiplication implied). Recall the meanings of the terms on the right-hand-side, which remain the same: the force of gravity acting on the parcel (that is, its weight); the force exerted by pressure on the bottom of the parcel; and the force exerted by pressure on the top of the parcel, respectively.

In our current example, we've applied Newton's Second Law to a parcel under the assumption that the three forces acting on it balance, and we've represented the forces in terms of other quantities (such as the pressure, density, acceleration due to gravity, dimensions of the parcel, etc.). We've represented the mass of the parcel as the product of its density, depth and cross-sectional area. To get the relation into a form with acceleration on the left-hand side and the forces per unit mass on the right-hand side, we divide by the mass represented as $m = \rho A \Delta z$:

$$a_z = 0 = -(\rho A \Delta z)g_0/(\rho A \Delta z) + p_{bot}A/(\rho A \Delta z) - p_{top}A/(\rho A \Delta z)$$

(where a_z is the acceleration of the parcel in the vertical direction). Dividing out the common factors in each term and rearranging the two pressure forces/mass, we can write this as:

$$a_z = 0 = -g_0 - [(p_{top} - p_{bot})/\Delta z]/\rho$$
$$= -g_0 - (\Delta_z p/\Delta z)/\rho$$

where we've defined the symbol $\Delta_z p \equiv p_{top} - p_{bot}$ to be the pressure difference in the vertical direction across the parcel.

   The quantity in parentheses, $\Delta_z p/\Delta z$, has dimensions of pressure over distance. It is the pressure difference in the vertical direction per unit of vertical distance, and it represents the rate at which pressure varies with respect to altitude. It is called the **vertical pressure gradient**.

   We've lumped the two pressure forces/mass together into one term, $-(\Delta_z p/\Delta z)/\rho$, which represents the net force per unit mass acting in the vertical direction due to pressure differences in the vertical direction. It is called the **vertical pressure-gradient force per unit mass**. In the atmosphere, pressure decreases with respect to altitude, so $p_{top} < p_{bot}$, which implies that $p_{top} - p_{bot} < 0$ and so $\Delta_z p < 0$ and so $-(\Delta_z p/\Delta z)/\rho > 0$, which tells us that vertical pressure-gradient force/mass is directed upwards, opposing the force/mass of gravity ($-g_0$), which is directed downwards. In our special case, this two upward and downward directed forces exactly balance each other, producing no vertical acceleration.

   The situation in which these two forces exactly balance each other is called **hydrostatic balance**. The equation above, which expresses this condition mathematically, is one form of the **hydrostatic equation**. In general, exact hydrostatic balance is rarely achieved—if it were, there would be no vertical motions at all! However, especially on large scales in the atmosphere and oceans, the force of gravity and the vertical pressure-gradient force almost or *approximately* balance, so we can make the **hydrostatic approximation** or **hydrostatic assumption** and still use the hydrostatic equation, with only small errors. However, on small scales—for example, inside a thunderstorm—there are sometimes large vertical accelerations produced by large imbalances between gravity and the vertical pressure-gradient force, and the hydrostatic assumption sometimes isn't valid.

   Also note that the hydrostatic equation is simply a special case of Newton's Second Law applied to the atmosphere (or the ocean) under the assumption of hydrostatic balance.

As an example of an application of the idea of approximate hydrostatic balance, consider a column of air extending from some arbitrary level in the atmosphere (for example, the surface of the earth) to "infinity". In that case, at the "top" of the column there is no air, and so nothing to exert pressure, and so the pressure at the top is zero. In this case, if we make the hydrostatic assumption, the force of gravity pulling down on the air in the column (that is, the weight of the column) is balanced solely by the upward force exerted by pressure at the bottom of the column:

$$0 = F_g + F_{Pbot}$$
$$= -(\text{weight of air in the column above}) + (P_{bot} \times A)$$

(where A is the area covered by the bottom of the column). Solving for the pressure at the bottom, we find that:

$$P_{bot} = (\text{weight of air in the column above})/A$$

where the right-hand side we interpret as the total weight of air in the column lying above *each unit* of area of the bottom of the column (that is, the weight per unit area).

Note that this relation does not *define* the pressure—the concept of pressure has nothing to do with gravity at all. Rather, it equates pressure quantitatively with the weight/area of air in the column above, under the assumption that hydrostatic balance prevails. (If the air were not in hydrostatic balance, the pressure would not equal the weight/area of the air above.)

To restate: under the hydrostatic approximation, pressure at any level in an imaginary column equals the total weight of air per unit area in the column above that level. (But the pressure is not the weight/area—it's just quantitatively *equal to* the weight/area.)