Relation between total and partial derivatives

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial s}$$, where

- *Q* is a field variable
- *c* is the speed of an observer/probe
- *s* is position along the probe's trajectory (that is, a 3-D "natural" coordinate)

Relation between total and partial derivatives

Derived by:

- Approximating total derivative using finite differences
- Manipulating the finite difference approximation algebraically
- Taking the limit as the time increment becomes infinitesimally small to get derivatives

Relation between total and partial derivatives

Interpretation:

- First, recall that total derivative is rate at which an observer observes, or probe measures, field variable *Q* to change w/r/t time as observer/probe moves through field of *Q*
- *Q* observed/measured in this way can change for two, separate reasons:
  - *Q* field is changing locally (that is, at fixed locations, *Q* is changing with time), so as you move from place to place you'd see changes in *Q*
    - True even if *Q* were spatially uniform
    - The rate of change you'd see depends on how fast Q is changing locally
  - *Q* field varies spatially, so moving from place to place you'd generally see changes in *Q*
    - True even if Q were steady
    - The rate of change you'd see depends on
      - (a) how fast you move through the field (your *speed*)
      - (b) how rapidly *Q* varies with position in the direction you're moving (the *gradient* of Q in that direction)

Relation between *material* and partial derivatives

Special case when $\vec{c} = \vec{U}$. That is, when we imagine that observer/probe ”follows” a fluid parcel, measuring Q for the same fluid parcel over time:

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + U\frac{\partial Q}{\partial s}$$

In this important special case:

- the total derivative becomes a material derivative, and
- the speed of observer becomes the speed of the parcel

Relation between *material* and partial derivatives

Solve for local derivative:

$$\frac{\partial Q}{\partial t} = -U\frac{\partial Q}{\partial s} + \frac{DQ}{Dt}$$

Interpetation:

The local rate of change in *Q* (left-hand side of the equation above) can be accounted for by a combination of two effects (right-hand side of the equation above):

- *Advection* of *Q* by the fluid (the parcel currently at your location leaves and is replaced by a parcel arriving from upstream or upwind, where *Q* might be different; or the arriving parcel transports Q from its original location to the location where you are monitoring Q)
- The arriving parcel itself might be experiencing changes in *Q* as it passes through your location

Coordinate systems

- All *vectors* comprise a set of two or more *scalar* pieces of information.
- To solve practical problems in fluid dynamics using mathematical relationships that involve vectors, we must represent vector quantities in terms of their *scalar components*.
    - It's the scalar components that are quantifiable—that is, they are the quantities to which we can assign numerical values
- For vectors that involve location in space (such as position, velocity, acceleration, and gradients of field variables), we must set up a *coordinate system* to define these components.

Some coordinate systems:

- **Rectangular/Cartesian** (2-D or 3-D):
    - Positions along the three axes are conventionally labeled x, y, and z
- **Cylindrical** (3-D):
    - Polar coordinates in 2-D [angle θ and radius R, in a plane] plus an axis normal to the polar plane [coordinate z])
- **Spherical** (3-D), such as the one used to identify location on Earth:
    - Latitude angle (φ);
    - longitude angle (θ); and
    - distance radially outward from center of the sphere (r) or from a reference radius (z)
- **Natural** (3-D) based on 3-D trajectory:
    - One axis (s) along the trajectory of an object;
    - One axis (n) normal to the s-axis and to its left; and
    - One axis (z) normal to the plane of s and n.
- **Natural** (3-D) based on projection of trajectory onto a horizontal (2D) plane:
    - One axis (s) along the horizontal trajectory of an object (the 3-D trajectory projected onto a horizontal plane);
    - One axis (n) normal to the s-axis and to its left, also horizontal; and
    - A vertical axis (z) normal to the horizontal plane of s and n.

Vector quantities in different coordinate systems

Examples of quantities involving vectors (or scalar products of vectors) in several coordinate systems:

Rectangular/Cartesian coordinates

- Position in space, $\vec{r}$:
  - $\vec{r}$ = (x,y,z) = $x\hat{\imath}$ + $y\hat{\jmath}$ + $z\hat{k}$
    (where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors—that is vectors with length = 1 and no dimensions—that point in directions parallel to the *x*-axis, *y*-axis, and *z*-axis, respectively)
    - *x*, *y*, and *z* are the scalar components of position
    - $x\hat{\imath}$, $y\hat{\jmath}$, and $z\hat{k}$ are the vector components of position

- Fluid parcel velocity, $\vec{U} \equiv \frac{D\vec{r}_p}{Dt}$:
  - $\vec{U} = (u,v,w) = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$
    - u, v, and w are the scalar components of fluid parcel velocity in the *x*-, *y*-, and *z*-coordinate directions, respectively; they can be positive or negative
    - $u\hat{\imath}$, $v\hat{\jmath}$, and $w\hat{k}$ are the vector components of fluid parcel velocity
    - $u \equiv \frac{Dx}{Dt}$, $v \equiv \frac{Dy}{Dt}$, and $w \equiv \frac{Dz}{Dt}$ (that is, u, v, and w are defined as the respective rates at which the parcel coordinates of position change w/r/t/ time)

- Gradient of field variable Q, ∇Q (a vector):
  - $\nabla Q = \frac{\partial Q}{\partial x}\hat{\imath} + \frac{\partial Q}{\partial y}\hat{\jmath} + \frac{\partial Q}{\partial z}\hat{k}$
    - $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial Q}{\partial z}$ are the scalar components of the gradient of Q in the *x*-, *y*-, and *z*-coordinate directions
    - $\frac{\partial Q}{\partial x}\hat{\imath}$, $\frac{\partial Q}{\partial y}\hat{\jmath}$, and $\frac{\partial Q}{\partial z}\hat{k}$ are the vector components of the Q gradient.

- Advection of Q (a scalar if Q is a scalar):
  - $-u\frac{\partial Q}{\partial x} - v\frac{\partial Q}{\partial y} - w\frac{\partial Q}{\partial z}$

Vector quantities in different coordinate systems

<u>Natural coordinates</u> (based on projection of a 3-D fluid parcel trajectory onto a horizontal plane)

- Position in space, $\vec{r}$:
  - $\vec{r}$ = (*s*,*n*,*z*) (where is is position along an axis comprising the projection of a 3-D parcel trajectory onto the horizontal plane)
  - *s*, *n*, and *z* are the scalar components of position

- Fluid parcel velocity, $\overrightarrow{U} \equiv \frac{D\vec{r}_p}{Dt}$
  - $\vec{U} = \vec{V} + w\hat{k} = (V, 0, w) = V\hat{s} + w\hat{k}$
    - V (the horizontal speed, which is always non-negative) and w (the vertical component of velocity, which can be positive or negative) are the scalar components of fluid parcel velocity in the horizontal (*s*- and *n*-coordinate plane) and vertical *(z*-coordinate) directions, respectively
    - $\vec{V} = V\hat{s}$ is the vector *horizontal* component of fluid parcel velocity and $w\hat{k}$ is the vector vertical component
    - $V \equiv \frac{Ds}{Dt}, and\ w \equiv \frac{Dz}{Dt}$ (that is, V and w are shorthand symbols for the scalar velocity components, which are defined as the respective rates at which the parcel coordinates of position change w/r/t/ time)

- Gradient of field variable Q, ∇Q*:*
  - $\nabla Q = \frac{\partial Q}{\partial s}\hat{s} + \frac{\partial Q}{\partial n}\hat{n} + \frac{\partial Q}{\partial z}\hat{k}$
    - $\frac{\partial Q}{\partial s}, \frac{\partial Q}{\partial n}$, and $\frac{\partial Q}{\partial z}$ are the scalar components of the gradient of Q in the *s*-, *n*-, and *z*-coordinate directions, respectively
    - $\frac{\partial Q}{\partial s}\hat{s}$, $\frac{\partial Q}{\partial n}\hat{n}$, and $\frac{\partial Q}{\partial z}\hat{k}$ are the vector components of the Q gradient.

- Advection of Q (a scalar if Q is a scalar):
  - $-V\frac{\partial Q}{\partial s} - w\frac{\partial Q}{\partial z}$