ERTH 430: Fluid Dynamics in Earth Systems

Lab #2: Scaling the Terms in the Governing Equations for Waves in a Wave Tank

(Fri. and Wed., Dec. 1 and 6)
(20 pts)

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I. Objectives

- Be able to scale (i.e., estimate the order of magnitude of) terms in the governing equations (tendency form) for waves in a water tank, based on simple measurements, on analytic modeling of some behavior, and on crude finite difference estimates of some derivatives in the equations.
- Be able to simplify the equations based in part on your scaling

II. Materials

- Water tank with wave generator
- Tracers (e.g., bits of crayon) of varying specific gravities (i.e., density normalized—that is, divided—by the density of water in the tank)
- Ruler
- Masking tape
- Stop watch or other timing device

III. Background

The governing equations are relatively complete* in the sense that they describe and constrain the behavior of an extraordinary range of fluid behaviors, from sound waves to turbulence, flow around obstacles, large scale ocean and atmospheric weather patterns, and every imaginable fluid flow in between. On the downside, there is no known general, analytic solution.

We can solve the governing equations approximately, though, by estimating the temporal and spatial derivatives using finite differences and solve the resulting (algebraic) equations on a computer. However, the resulting solutions will have errors (at least partly because the derivatives in the equations and the initial and boundary conditions aren't exact). Moreover, it's not always easy to understand which physical processes are important for creating and controlling any particular phenomenon because all processes are represented in the solution.

* Supplemented as needed by one additional equation for the concentration of each constituent of the fluid that might vary in response to sources and sinks of the constituent. Examples include dissolved salt in sea water, and water in different phases (vapor, liquid, ice) in the atmosphere.

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Another approach to solving the governing equations is to "scale" the terms in them as they apply to the particular phenomenon of interest (that is, estimate the characteristic size of the terms to the nearest order of magnitude as they apply to a particular phenomenon), then neglect all terms except the largest, then solve the resulting, simplified equations.

In this lab, you'll conduct a partial scaling exercise like this, for periodically forced waves on water in the classroom wave tank.

IV. Instructions

(1) [4.5 pts total] <u>Measurements of wave behavior</u>

With the water in the tank at rest (that is, not moving), select a tracer that floats and another that sinks. Drop them into the tank at the same place.

(*i*) [0.5 pts] *What is the depth of the undisturbed water?*

(*ii*) [1 pt] Turn on the water tank's wave generator and set its frequency setting to "5". *What is the period of oscillation of the generator?*

Let the waves develop in the tank until they appear to be regular, periodic features. Then observe the behavior of the tracers at the top and bottom of the water layer (and in between if you have a tracer suspended there)**.** At the water surface, confirm that they move in nearly closed, looping trajectories, and at the bottom of the tank confirm that they move back and forth horizontally. (What about in between the top and bottom of the water?)

(*iii*) [2 pts] Assuming that the tracers move in the same way as "parcels" of water in the tank, *estimate the vertical amplitude, horizontal amplitude, and period of the oscillatory motion of water parcels at the surface. Do the same for water parcels at the bottom.*

(*iv*) [1 pt] *Estimate the wavelength of waves in the tank.* (Do you think that the wavelength of the disturbance at the bottom of the tank is any different from the wavelength of waves at the surface?)

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(2) [7.5 pts total] Lagrangian modeling of the motions of water parcels in waves

(a) [5.5 pts total] The oscillatory, looping behavior of tracers suggests that we might model the vertical and horizontal position of water parcels expressed in rectangular coordinates, $x_p(t)$ and $z_p(t)$, at least approximately using sine and cosine functions, which are well known periodic functions:

$$z_p(t) = A_z(d)\cos(2\pi t/T)$$

$$x_p(t) = -A_x(d)\sin(2\pi t/T)$$

where:

- T is the period of oscillation;
- $A_z(d)$ and $A_x(d)$ are the half-amplitudes of the oscillations in the vertical and in the horizontal (along-tank) directions, respectively; and
- d is the parcel's average vertical distance from the bottom of the tank.

In this case, the origin of the rectangular coordinate system is at the center of the (modeled) parcel's elliptical or circular trajectory, and the x-axis is oriented to the right as you face the tank.

The time, t, begins (that is, $t = 0$) when the parcel is at the peak of a wave (where $z_p(0) = A_z(d)$, which you can confirm by substituting $t = 0$ into the expression for $z_p(t)$ above).

(*i*) [0.5 pts] The half-amplitudes (especially the vertical half amplitude) might vary with depth in the tank. *What is the vertical half-amplitude at the bottom of the tank ($A_z(0)$)? What about the horizontal half-amplitude at the bottom ($A_x(0)$)?*

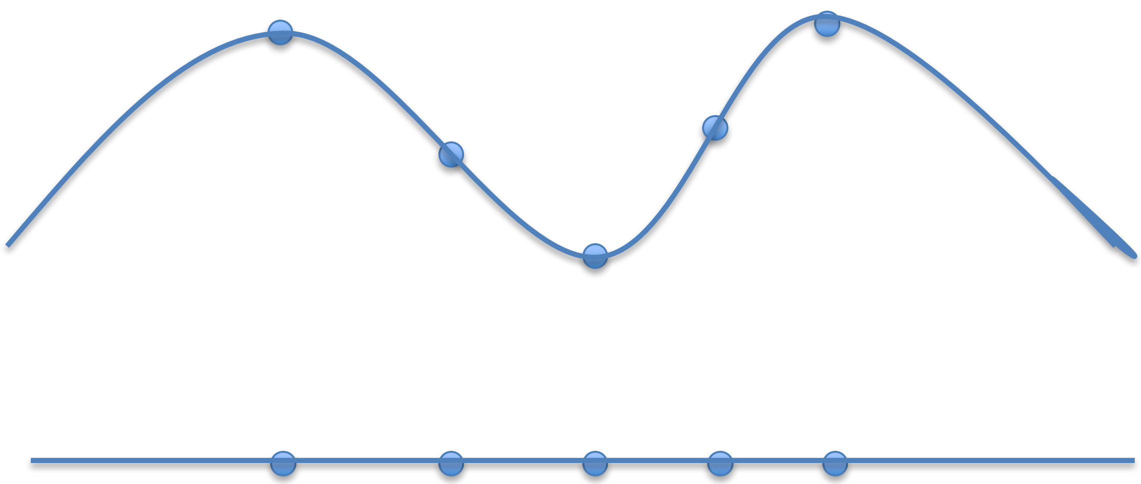
(*ii*) [1 pt] According to this model, a water parcel's position over time should describe a circle (if $A_z(d) = A_x(d)$) or an ellipse (if $A_z(d) \neq A_x(d)$), consistent with the behavior of tracers that you observed as waves passed by. *According to the model above, does a parcel move clockwise or counterclockwise? How can you tell? Is this consistent with what you observed?*

(*iii*) [2 pts] Assuming that this model is at least roughly accurate, *estimate the parcel's maximum vertical velocity component* ($w(t) \equiv Dz_p(t)/Dt$) *and the maximum horizontal (that is, along-tank) component of its velocity* ($u(t) \equiv Dx_p(t)/Dt$) *at both the surface and bottom of the tank*.

(*iv*) [2 pts] Similarly, *estimate the maximum vertical and horizontal components of the parcel's acceleration* ($a_z(t) \equiv Dw/Dt$ *and* $a_x(t) \equiv Du/Dt$, respectively).

(b) [2 pts total]

(*i*) [1.5 pts] *On the diagram below, at each designated spot (the 10 dots), draw separate, labeled arrows to indicate the direction of horizontal (u) and vertical (w) components, respectively, of the motion of a water parcel at that spot at the moment shown.* (The wave disturbance travels to the left, as you'd view it in the wave tank in our classroom.)



(*ii*) [0.5 pt] Assuming that our model for $z_p(t)$ and $x_p(t)$ is reasonable at each of the 10 parcels at the spots indicated, *what normalized" time, t/T, between 0 and 1 (that is, what t as a fraction of T), would have to apply at each spot for the model to represent the appropriate signs of both velocity components there?*

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(3) [8 pts total] Eulerian modeling of the motions of water in waves

The Lagrangian model in **(2)(a)** might apply (approximately) to many individual parcels in the wave pattern separately, but we would have to modify the model to describe all of their positions at any given time because parcels at different horizontal locations in the wave are at different parts (phases) of their elliptical orbits, as your answer to **(2)(b)** should show. To account for such spatial variations in the phase of parcel motions, let's assume that the following model is a reasonable approximation of the fields of u- and w-components of parcel motion. The model describes how u and w vary not only with respect to time (t) and vertical position (which we'll relabel here as z), but also with position along the tank (x, which we'll define to increase to the right when we face the tank):

$$u(x, z, t) = -A_x(z)(2\pi/T)\cos\left(\frac{2\pi}{L}(x + ct)\right)$$

$$w(x, z, t) = -A_z(z)(2\pi/T)\sin\left(\frac{2\pi}{L}(x + ct)\right)$$

where:

- L is the wavelength of the waves, and
- c is the "propagation speed" of the waves (*not* the speed of the fluid parcels—the wave is a pattern, not an object, so the speed of the pattern and the speed of physical bits of matter participating in the pattern are not necessarily the same).

The following relationship applies:

$$c = L/T$$

That is, any particular part of the wave pattern travels a distance equal to the wavelength of the pattern in the amount of time equal to the period of the wave pattern. The *phase* of the periodic pattern, which determines relative position within the pattern expressed as an angle in radians, is $(2\pi/L)(x + ct)$. In this model, x = 0 at the peak of one of the waves at t = 0.

(i) [1 pt] *Does this model of the spatial and temporal pattern of u and w agree with your version of the figure in* ***(2)(b)****? Explain.*

(*ii*) [3 pts] Refer to the tendency form of the governing equations, in particular to the velocity tendency equations for *u* and *w*. Using the model above for the fields of *u* and *w* in the water, *write the velocity component tendencies (* $\partial w/\partial t$ and $\partial u/\partial t$ *) and the velocity-component advection terms that matter here (* $-u\,\partial w/\partial x$ and $-w\,\partial w/\partial z$ in the *w*-tendency equation, and $-u\,\partial u/\partial x$ and $-w\,\partial u/\partial z$ in the *u*-tendency equation*) as explicit functions of x, z, and t.*

To do this, for simplicity assume that the half-amplitudes of parcel horizontal and vertical motion, $A_x(z)$ and $A_z(z)$, respectively, vary linearly with *z*. This is probably reasonable except in a very shallow layer immediately next to the bottom of the tank. Perhaps the most intuitive way to write $A_z(z)$ as a linear function of *z* might be:

$$A_z(z) = A_z(0) + \left(A_z(\bar{h}) - A_z(0)\right)\left(z/\bar{h}\right)$$

where $\bar{h}$ is the average depth of the water and *z* is vertical position in the water, measured upward from the bottom of the tank, when there are no waves in the tank. Check to make sure that this makes sense at the top and bottom of the water by checking the value that this formula gives us for $A_z(z)$ when z = 0 and when z = $\bar{h}$. (Use a similar linear function to describe $A_x(z)$.)

(*iii*) [1 pt] *Write the force/mass of friction in the u- and w-tendency equations as explicit functions of x, z, and t*, assuming that the linear models for $A_x(z)$ and $A_z(z)$ in (*ii*) above apply. (Again, this is probably not a bad assumption except in a very shallow layer next to the bottom of the tank, which we'll ignore here.)

(*iv*) [2 pts] *Estimate the maximum magnitude of each term in (ii) and (iii), considered over all values of t, x*, and *z* within the water.

(*v*) [1 pt] Based on your results, *what can you say about the characteristic magnitude of the horizontal and vertical pressure gradients, respectively?*