ERTH 430: Fluid Dynamics

In Earth Systems

Force Per Unit Mass

of Friction in Fluids

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Summary. *Shear* is the gradient of velocity in a direction normal to the velocity. In the presence of shear, collisions among molecules in random motion tend to transfer momentum "down-shear" (from faster toward slower speeds). This can change parcel velocities; it constitutes a force of friction. The greater the shear, the faster momentum will transfer down-shear.

The *net* rate at which this transfer of molecular momentum into and out of a fluid parcel acts to change the momentum of the parcel depends on how rapidly this molecular momentum transfer varies from one side of the parcel to the other, and hence on the gradient across the parcel of the shear.

As an example, in a rectangular coordinate system, the *x*-component of the force per unit mass of friction can be written as:

$$\left(F_{fr}\right)_x = \nu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $\nu \equiv \mu/\rho$ is the *kinematic viscosity* of the fluid and μ is the *dynamic viscosity*, a measure of the extent to which a fluid resists deformation and flowing. (Analogous expressions apply to the other two components of friction/mass.)

The relation between force of friction/mass and curvature in velocity profiles.

Consider a fluid parcel (which might be moving) that has neighboring fluid moving past the top of it (that is, the parcel's velocity differs from the velocity of adjacent fluid above it). The parcel and the adjacent fluid "rub" or "drag" across their mutual interface, each exerting a force (friction) on the other (in opposite directions) parallel to the interface. Why does this happen, and how can we express mathematically the transfer of momentum across the interface that friction implies?

For convenience, suppose that the interface in question is horizontal and we adopt a rectangular coordinate system in which we orient the *z*-axis vertically and the *x*- and *y*-axes horizontally. In this case, the *z*-component of molecular motions is perpendicular to the interface and the *x*- and *y*-components are parallel to it.

On the molecular scale, the force of friction is, like the force due to pressure, mediated by collisions between molecules in random motion at the interface between the parcel and the adjacent fluid. The randomness of molecular motions mean that vast majority of molecules typically move at some angle or another relative to the fluid interface, not exactly perpendicular or exactly parallel to it. Hence, the velocity of any particular molecule comprises a part normal to the interface (the *z*-component) and a part parallel to the interface (the *x*- and *y*-components combined). The part normal to the interface (in particular, motion

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toward the interface rather than away from it) leads to collisions with molecules of fluid on the other side of the interface, thereby exerting a force on the fluid there. The collective z-component of the force of many such molecular collisions, per unit area of the interface, is what we call *pressure*. The force due to pressure is directed normal to the interface (in the z-direction, in our example).

To restate: fluid on each side of a horizontal interface exert pressure on each other because the z-component of molecular motions leads to molecular collisions across the interface.

The velocity of a molecule's motion also includes components (the x- and y-components) *parallel* to the interface. When two molecules collide at the interface (thanks to their z-component of motion toward the interface), the molecule with the faster x-component of motion tends, on the average, to transfer some of it to the molecule with the slower x-component of motion, so the respective x-components end up more similar than before the collision. (More colloquially, the faster-moving one tends to slow down and the slower one tends to speed up, making their speeds more similar.) The collisions conserve the total (that is, sum of the) x-components of momentum of the two molecules, while (on the average) making them more similar. The same is true of the y-components of molecular motion.

If the *average x-component* and y-component of molecular motions is the *same* on *each side* of the interface, then, averaged over many collisions at the interface, there will be no net change in the x- or y-component of molecular motions on either side of the interface. That's because on each side of the interface, there will be just as many molecules that speed up as slow down as a result of collisions. Another way of saying this is that in this case, there is no net transfer of x- or y-component of molecular momentum across the interface, and so the fluid on each side of the interface won't exert a force friction on the other.

However, if the average x- and/or y-component of molecular motions *differ* across the interface, then, averaged over many collisions, the average x- and/or y-components of molecular motions on each side of the interface will end up more similar than before the collisions. That is, the collisions result in a net transfer of x- and/or y-component of momentum across the interface from the faster fluid to the slower fluid. Since the momentum (in the x- and/or y-direction) of the fluid on each side of the interface changes, this means that there must be a force exerted in the x- and/or y-direction on the fluid on each side of the interface (in opposite directions).

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This is the force of friction. It is the net transfer across a fluid interface of the component of momentum parallel to the interface, as a result of collisions at the interface of molecules in random motion. *Friction occurs when the average component of molecular motions parallel to the fluid interface differs across the interface.*

The average velocity of molecular motions in a parcel of fluid is just the parcel *velocity*. (The motions of molecules relative to the average are random.) We can rephrase the condition under which a fluid parcel will experience a force of friction as follows: *A fluid parcel experiences a force due to friction when the parcel moves past adjacent fluid or vice versa.* In this case, the component of fluid velocity parallel to the interface between the parcel and the surrounding fluid varies across the interface.

Intuitively, the greater the difference there is across an interface in the component of velocity parallel to the interface, the stronger the force of friction should be. However, in real fluids, velocity doesn't vary abruptly (discontinuously) across an interface but rather continuously. Hence, we consider not the difference in the interface-parallel component velocity across an interface (in our example, Δu and Δv) but rather the *gradient* across the interface ($\partial u/\partial z$ and/or $\partial v/\partial z$).

In general, the gradient of velocity in a direction normal to the velocity itself is called *shear*. Specific cases include the vertical gradient of the horizontal velocity (that is, *vertical shear of the horizontal velocity*), which is a vector with x- and y-components of $\partial u/\partial z$ and $\partial v/\partial z$, respectively. The horizontal gradient of the horizontal velocity (*horizontal shear of the horizontal velocity*) is a vector with horizontal components in rectangular coordinates of $\partial u/\partial y$ and $\partial v/\partial x$.

Hence, intuitively, the x- and y-components of friction on a horizontal fluid interface should be proportional to $\partial u/\partial z$ and $\partial v/\partial z$, respectively. It should also be proportional to the *area* of the interface (A), because the larger the interface between two fluids moving relative to each other, the more molecules there will be colliding and transferring momentum across the interface, and so the greater the total rate of momentum transfer (that is, the total force of friction) should be.

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     Consider a rectangular parallelepiped fluid parcel with horizontal top and bottom and vertical sides. The bottom is located at z-coordinate $= z$ and the top at $z + \Delta z$ (**Figure 1**). Based on the foregoing, at the top of the fluid parcel we can write:

$$(1)(a) \quad \left(F_{fr}(z + \Delta z) \right)_x = \mu A \frac{\partial u(z + \Delta z)}{\partial z}$$

where:

- $\left(F_{fr}(z + \Delta z) \right)_x$ is the x-component of the force of friction acting on the top of the fluid parcel;
- A is the area of the interface between the top of the parcel and the adjacent fluid above it;
- $\partial u(z + \Delta z)/\partial z$ is the vertical shear of the u-component of fluid velocity at interface; and
- μ is a (positive) constant of proportionality. It is called the *dynamic viscosity* of the fluid.

     You can verify that μ (dynamic viscosity) necessarily has dimensions of (force/area)×time, or pressure×time, or [(momentum/time)/area]×time (i.e., momentum flux×time). Dynamic viscosity depends on the nature of the fluid, so, for example, water and air have very different dynamic viscosities. You can think of it as a measure of the fluid's resistance to deformation or flowing.

     The sign of the right-hand side of the Eq. (1)(a) above is correct because when $\partial u(z + \Delta z)/\partial z$ is positive (that is, the u-component of velocity increases upward, assuming that z increases upward), the x-component of momentum will be transferred downward (from the greater u-component toward the lesser u-component), which at the top of the parcel is into the parcel. This amounts to a force (x-component of friction) in the positive x-direction.

     In contrast, at the bottom of the parcel:

$$(1)(b) \quad \left(F_{fr}(z) \right)_x = -\mu A \frac{\partial u(z)}{\partial z}$$

(For simplicity, we assume here that μ is spatially uniform, which is not entirely true in reality.) Again, if $\partial u(z)/\partial z$ is positive, then the x-component of momentum will be transferred downward, but at the bottom of the parcel this is out of the parcel, which amounts to a force (x-component of friction) on the parcel in the negative x-direction, which requires that we include the minus sign explicitly.

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As an aside, at the interface we can define the *viscous shear stress,* the force of friction per unit area at a fluid interface:

$$(2) \quad \tau_x \equiv \left(F_{fr}\right)_x / A = \mu \frac{\partial u}{\partial z}$$

The *x*-component of viscous shear stress (τ_x) is proportional to the vertical shear of the *x*-component of velocity across that interface, $\partial u/\partial z$. It has dimensions of force/area, or pressure, or (momentum/time)/area (i.e., momentum flux). (Note that it is not physically a pressure, but it has the same dimensions as pressure and shares some things in common, including an interpretation as the flux of momentum due to collisions of molecules in random motion. In the case of pressure, the component of momentum "fluxing" across the surface of the parcel is normal to the surface, while for viscous shear streass it is the component of momentum parallel to the surface.

We are usually interested not in the force acting on a parcel but the force per unit pass, since that is what determines the acceleration. In our example, the *net* force of friction in the *x*-direction per unit mass due to friction acting at both the top and bottom of the parcel is sum of the forces of friction at the top and bottom of the parcel (Eqs. (1)(a) and (b)) divided by the mass of the parcel. We can relate the mass of the parcel to its average density times its volume, and relate its volume to the surface area at the top and bottom (A_z) and the height of the parcel (Δz) (**Figure 1**):

Net $\left(F_{fr}\right)_x$ per unit mass on top and bottom

$$= \frac{\left(F_{fr}\right)_x (z+\Delta z) - \left(F_{fr}\right)_x (z)}{m}$$

$$= \frac{A\mu \dfrac{\partial u(z+\Delta z)}{\partial z} - A\mu \dfrac{\partial u(z)}{\partial z}}{\rho A \Delta z} = \left(\frac{\mu}{\rho}\right)\left(\frac{\dfrac{\partial u(z+\Delta z)}{\partial z} - \dfrac{\partial u(z)}{\partial z}}{\Delta z}\right)$$

In the limit as the size of the parcel becomes infinitesimally small, this becomes:

(3) Net $\left(F_{fr}\right)_x$ per unit mass on top and bottom

$$= \nu \frac{\partial(\partial u(z)/\partial z)}{\partial z} \equiv \nu \frac{\partial^2 u(z)}{\partial z^2}$$

where $\nu \equiv \mu/\rho$ is the *kinematic viscosity*. (It has dimensions of (force/mass)×distance×time, or distance×velocity, or area/time.)

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Eq. (3) tells us that the net force/mass due to friction at the top and bottom of the parcel is proportional to the $\frac{\partial^2 u(z)}{\partial z^2}$, the *curvature* in the vertical profile of $u(z)$. Since the vertical flux of momentum is proportional to the vertical shear in the horizontal velocity, vertical variations in the vertical shear (that is, non-zero vertical curvature in the horizontal wind) imply that the flux of momentum associated with friction at the top and bottom of the parcel are not equal. As a consequence, the parcel gains and loses momentum through the top and bottom at different rates, creating a net gain or loss of momentum and hence a net force due to friction.

There is a similar contribution to the net frictional force the *x*-direction due to friction on the front and back of the parcel. Taking this into account gives the more complete statement:

(4)(a)   Net $\left(F_{fr}\right)_x/m$ on the top, bottom, front, and back

$$= \nu \frac{\partial^2 u}{\partial z^2} + \nu \frac{\partial^2 u}{\partial y^2} = \nu \left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Analogously,

(4)(b) Net $\left(F_{fr}\right)_y/m$ on the top, bottom, left face, and right face

$$= \nu \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2}\right)$$

(4)(c)   Net $\left(F_{fr}\right)_z/m$ on the front, back, left face, and right face

$$= \nu \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right)$$

(Again, Eqs. (4) assume that the dynamic viscosity, μ, is uniform in the fluid, which is not strictly true, but it might be approximately true in most cases.)

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Figure 1: A rectangular parallelepiped fluid parcel in an environment with positive vertical shear in the *u*-component of velocity at the top and bottom of the parcel.

u-component of fluid velocity just above the top of the parcel

Area of top and bottom of the parcel $\equiv A_z$

$z + \Delta z$

u-component of fluid velocity just below the top of the parcel

u-component of fluid velocity just above the bottom of the parcel

$y + \Delta y$

z

u-component of fluid velocity just below the bottom of the parcel

y

x

$x + \Delta x$

Rectangular coordinate system