ERTH 430: Fluid Dynamics in Earth Systems	**In-Class Exercise: Time Derivatives, Illustrated**	Dr. Dave Dempsey Department of Earth & Climate Sciences, SFSU

Introduction

Figures 1.1, **1.2**, and **1.3** show three fluid parcels, labeled P_1, P_2, and P_3, at two different times, t_0 and t_1 (where $t_1 > t_0$). At each time, a *Q*-probe, which measures the value of some physical property, *Q*, of the fluid, at wherever the probe is located, is also shown. (Although not all field variables are properties of fluids, properties of fluids such as air in the atmosphere, water in the ocean, streams, lakes, and underground aquifers, are typically field variables because they vary continuously with respect to position and time within the fluid, so we can write *Q* as $Q(\vec{r}, t)$—that is, *Q* depends (continuously) on location, $\vec{r}$, and time, *t*.

Generically speaking, position and time are independent of each other—that is, they are *independent variables*—and field variables *depend on* them, or are *functions of* them, which is what we mean when we write $Q(\vec{r}, t)$. That is, given a location within the field and a time (chosen independently of each other), *Q* has some (numerical) value there.

However, it's also possible to talk about the position of some specific object, such as a fluid parcel, a *Q*-probe, a baseball, or whatever. Objects can move, in which case their positions vary with time—that is, the location of a specific object depends on, or is a function of, time. In that case we'd write the location of the object something like $\vec{r}_{obj}(t)$. That is, we put a subscript on the more generic symbol for location, $\vec{r}$, to indicate that it refers to the location of *some specific thing (object)*, and make it clear that this location can depend on (vary with) time. This distinction between generic location in space, $\vec{r}$, which is *independent* of time, and $\vec{r}_{obj}(t)$, the location of some specific object, which can *depend on* time, is an important one to make.

By definition, the *velocity* of an object is the rate at which the object's position varies with respect to time $\equiv D\vec{r}_{obj}(t)/Dt$. (Note that because position in space is a vector, velocity must be, too.)

All three parcels and the *Q*-probe in **Figures 1.1**–**1.3** could in principle be moving. The velocity of the fluid, $\vec{U}(\vec{r}, t)$, is a field variable and so can be expressed as a function of location and time (treated independently of each other). In **Figures 1.1**–**1.3**, the velocity of each of the three fluid parcels at t_0 is shown as a thick arrow and labeled. (The length of each arrow is proportional to the speed of the parcel.) One relatively explicit way to write the velocity of a particular fluid parcel is $\vec{U}\left(\vec{r}_p(t), t\right) \equiv D\vec{r}_p(t)/Dt$, where $\vec{r}_p(t)$ is the location (position) of the fluid parcel. This is the notation used in **Figures 1.1**–**1.3**.

Unlike a typical fluid in the Earth system, the *Q*-probe is a discrete object that is *not* distributed over a relatively large region in space. As a consequence, its velocity, which we choose to write using a different symbol, $\vec{c}(t)$, isn't a continuous function of location and therefore isn't a field variable, but it is a function of time. The *Q*-probe's velocity at t_0 is shown and labeled in the figures.

Instructions

(1) Below is a list of *Q* values written to make the location and time at which each one applies explicit enough to locate on **Figures 1.1**–**1.3**. On each figure, write each of the *Q* values so that it is clear where (that is, at what location) it applies. For this purpose, the location/position of the *Q*-probe is written $\vec{r}_{obs}(t)$.

a) $Q(\vec{r}_{P_1}(t_0), t_0)$
b) $Q(\vec{r}_{P_1}(t_1), t_1)$
c) $Q(\vec{r}_{P_2}(t_0), t_0)$
d) $Q(\vec{r}_{P_2}(t_1), t_1)$
e) $Q(\vec{r}_{P_3}(t_0), t_0)$
f) $Q(\vec{r}_{P_3}(t_1), t_1)$
g) $Q(\vec{r}_{obs}(t_0), t_0)$
h) $Q(\vec{r}_{obs}(t_1), t_1)$
i) $Q(\vec{r}, t_0)$ (where $\vec{r}$ is the place where the *Q*-probe and parcel P_1 start out and parcel P_2 ends up)
j) $Q(\vec{r}, t_1)$ (where $\vec{r}$ is the same as in (i) above)

(2) For each diagram, use appropriate values on the list above to construct a finite difference estimate of each of the following time derivatives. (As an aside, note that your finite difference estimate of the derivative over the interval between t_0 and t_1 will equal the derivative exactly at some *t* in the range between t_0 and t_1. It is also equal to the average value of the derivative over that interval.)

a) $\frac{DQ(\vec{r}_{p2}(t),t)}{Dt}$
b) $\frac{DQ(\vec{r}_{p1}(t),t)}{Dt}$
c) $\frac{dQ(\vec{r}_{obs}(t),t)}{dt}$
d) $\frac{\partial Q(\vec{r},t)}{\partial t}$ (where $\vec{r}$ is the same as in **(1)**(i) above)

(3) For each of the following conditions, identify the figure that illustrates the scenario for which the condition is true in general (that is, not just for special patterns of Q). If no figure matches, say "no match".

a) $\frac{DQ(\vec{r}_{p_1}(t),t)}{Dt} = \frac{dQ(\vec{r}_{obs}(t),t)}{dt}$

b) $\frac{\partial Q(\vec{r},t)}{\partial t} = \frac{dQ(\vec{r}_{obs}(t),t)}{dt}$

c) $\frac{\partial Q(\vec{r},t)}{\partial t} = \frac{DQ(\vec{r}_{p_1}(t),t)}{Dt}$

d) $\frac{DQ(\vec{r}_{p_1}(t),t)}{Dt}$, $\frac{\partial Q(\vec{r},t)}{\partial t}$, and $\frac{dQ(\vec{r}_{obs}(t),t)}{dt}$ are all potentially different

(4) Repeat **(3),** but suppose that the Q field is *uniform* and *stays uniform over time* (but is not necessarily *steady*).

(5) Repeat **(3)**, but suppose that the Q field is *steady* (but not *uniform*).

(6) In practice, the local (Eulerian) and total derivatives are easy to estimate (by finite difference methods) because we measure/observe the properties of fluids using instruments that aren't attached to fluid parcels but instead are fixed in location (e.g., weather stations and stream flow gauges) or move through the fluid at arbitrary velocities that don't happen to match the fluid velocity (e.g., instruments attached to planes, boats, and cars). However, the fundamental principles that govern or constrain the behavior of fluids are mostly conservation laws, which (as we will see) involve the material (or parcel or Lagrangian) derivative. Hence, to make use of both the governing principles and actual observations to help us understand and predict the behavior of fluids (and we need both), we need a relationship between total, material, and local derivatives. To derive such a relationship, refer to **Figure 2**.

(a) For this derivation, we'll adopt a "natural" coordinate system, which has the following characteristics:

- The origin as at some point of interest.
- One coordinate direction is along the trajectory of whatever object is located at the point of interest (which is in the direction of the object's velocity). The position along this axis is often called "*s*".
- A second coordinate direction is normal to (that is, perpendicular to) the *s*-axis and to the left of it (when facing in the direction of the first one). The position along this axis is typically called "*n*".

- The third coordinate direction is normal to both of those. The position along this axis is typically called "*z*", and it is typically chosen to point vertically upward, in which case the *s*- and *n*-axes lie in a horizontal plane. (In this case, only the horizontal part of the 3-dimensional velocity is used to determine the two coordinate directions in the horizontal plane).

Draw the trajectory of the *Q*-probe from t_0 to t_1. This is part of the *s*-axis of a natural coordinate system that is based on the trajectory of the *Q*-probe. Label the coordinates of the *Q*-probe along the *s*-axis $s_{obs}(t_0)$ and $s_{obs}(t_1)$ at times t_0 and t_1, respectively.

(b) Write a finite-difference approximation of dQ/dt between the times t_0 and t_1.

(c) Set your expression in **(b)** equal to the same thing except with $-Q(\vec{r}_{obs}(t_1), t_0) + Q(\vec{r}_{obs}(t_1), t_0)$ inserted between the two terms in the numerator. (Of course, this is equal to 0 and so doesn't change the values to which it is added, thereby preserving the equality.) The result should have four terms in the numerator.

(Note that two different times--$t_1$and t_0--appear in $Q(\vec{r}_{obs}(t_1), t_0)$. What does this mean? That is, at what location and time does this value of *Q* apply?)

(d) Rewrite your result from **(c)** by splitting it into two separate terms. The first term should contain the first two terms in the numerator and the second term should contain the third and fourth terms in the numerator. (Of course, the denominator of each term is the same.)

(e) Set your result in **(d)** equal to the same thing but with the second term multiplied by $\left(s_{obs}(t_1) - s_{obs}(t_0)\right)/\left(s_{obs}(t_1) - s_{obs}(t_0)\right)$. (Place this factor in front of the second term. This factor is, of course, equal to 1, so it doesn't change the value of the quantity it multiplies, thereby preserving the equality.)

(f) Set your result in **(e)** equal to the same thing except with the denominators of the two factors in the second term on the right-hand side swapped.

(g) Evaluate the original left-hand side of the your results from **(b)** through **(f)**) in the limit as $t_1 \rightarrow t_0$. What type of derivative is it? Where (in space and time) does it apply? Write it using standard short-hand notation for this type of derivative.

Evaluate the left-hand term on the right-hand side of your result in **(f)** in the limit as $t_1 \rightarrow t_0$. What type of derivative is it? Where (in space and time) does it apply? Write it using standard short-hand notation for this type of derivative.

Evaluate the first of the two factors in the second term on the right-hand side of your result in **(f)** in the limit as $t_1 \rightarrow t_0$. What type of derivative is this? Where (in space and time) does it apply? Write it using standard short-hand notation for this type of derivative. There is an even shorter-hand notation for this derivative; write it using that shorter notation.

Evaluate the second of the two factors in the second term on the right-hand side of your result in **(f)** in the limit as $t_1 \rightarrow t_0$. What type of derivative is this? Where (in space and time) does it apply? Write it using standard short-hand notation for this type of derivative.

(h) Write the whole relationship using standard short-hand notation for each type of derivative.

(i) Suppose that the velocity of the *Q*-probe at t_0, $\vec{c}(t_0)$, is equal to the velocity, $\vec{U}(\vec{r}_p(t_0), t_0)$, of whatever fluid parcel is at the *Q*-probe's location at t_0 (so that $\vec{r}_p(t_0) = \vec{r}_{obs}(t_0)$). How would the relationship in **(h)** change? (Rewrite it to reflect all of the changes. *Hint*: There are two notational changes.)

Figure 1.1: Time derivatives, illustrated.

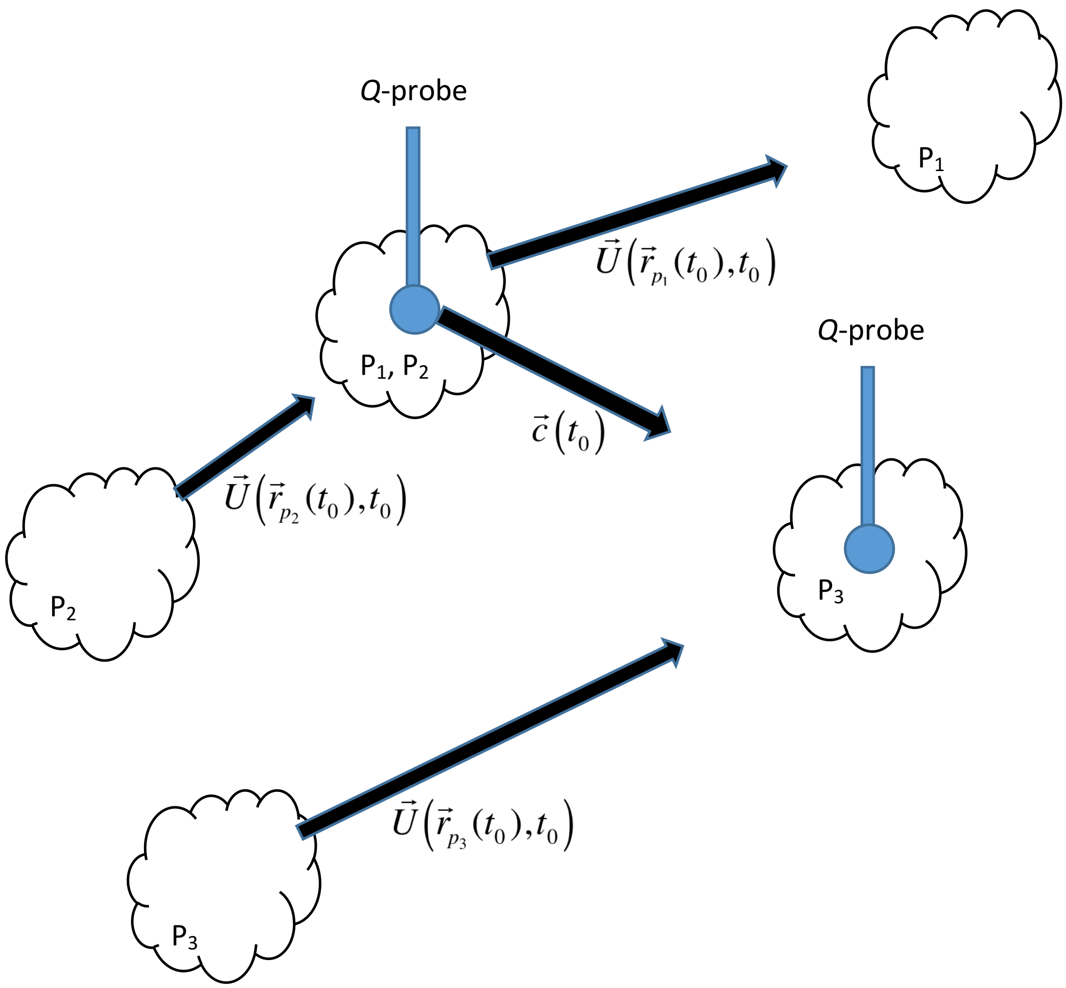

Figure 1.2: Time derivatives, illustrated.

As in **Figure 1.1**, except that the velocity of the *Q*-probe is different, and so the *Q*-probe's location at t_1 is also different.

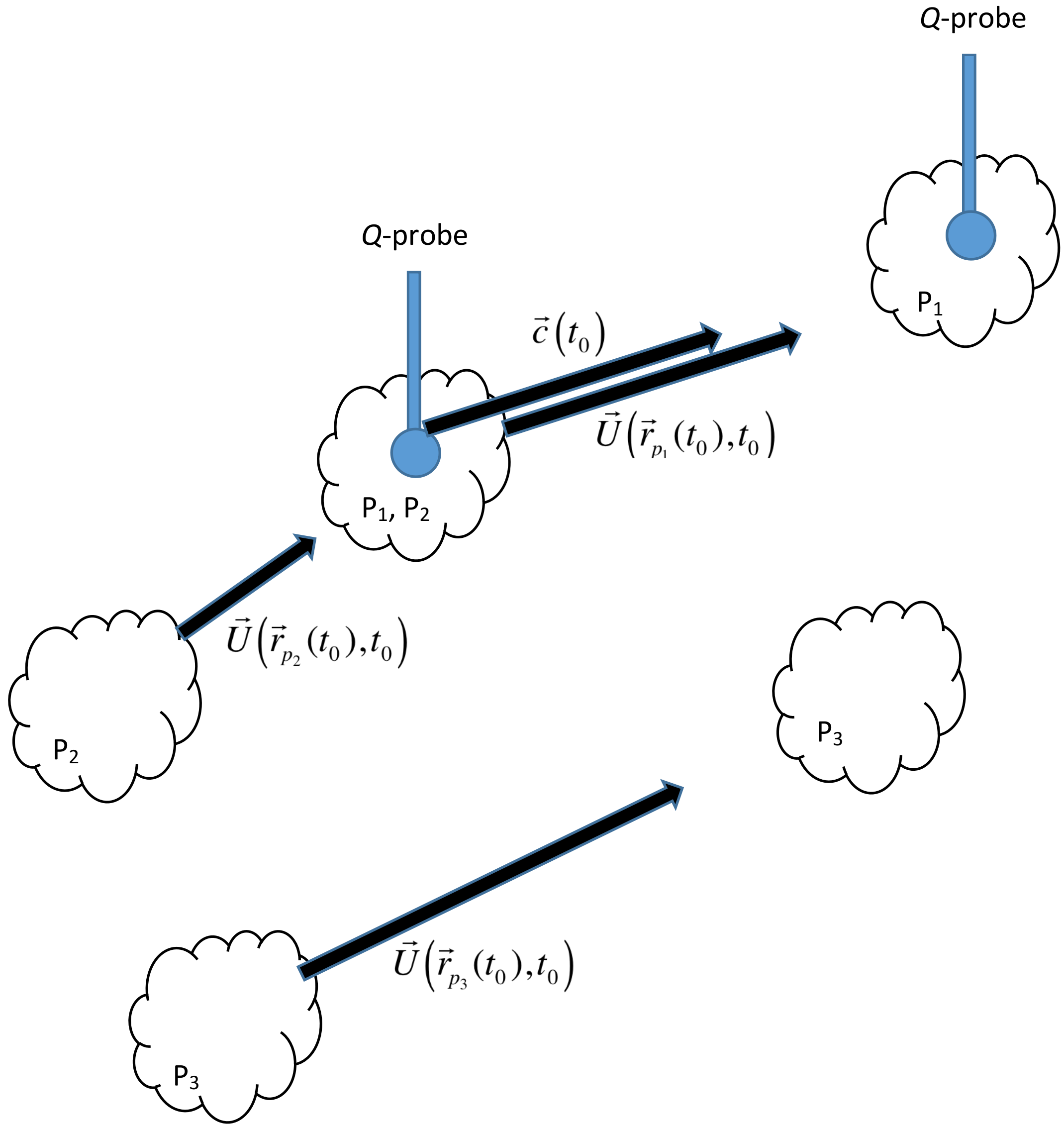

Figure 1.3: Time derivatives, illustrated.

As in **Figures 1.1** and **1.2**, except that the velocity of the *Q*-probe is different from the other two cases, and hence the Q-probe's location at t_1 is also different.

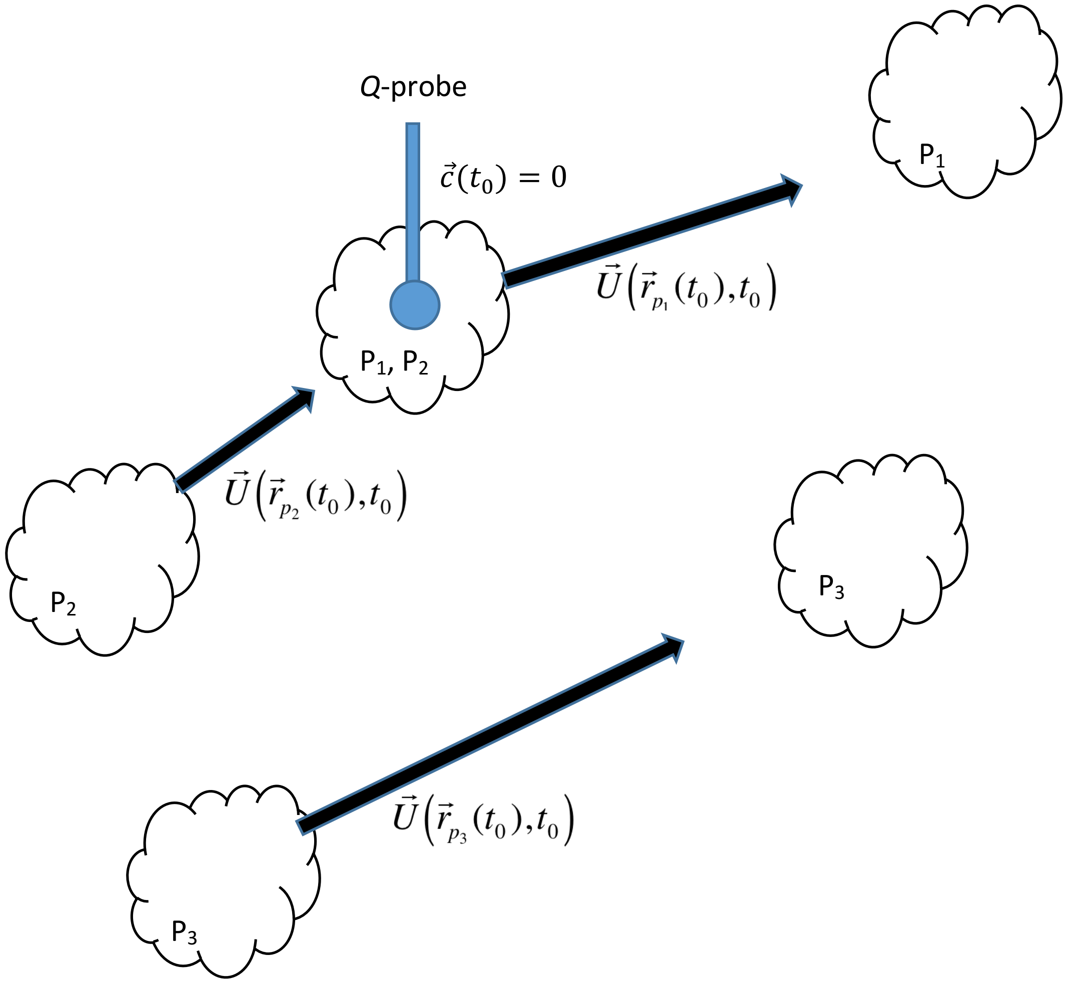

Figure 2. Relationship between total and partial time derivatives, (partly) illustrated. The notational conventions are the same as in **Figures 1.1–1.3** (but the parcel labels and velocities are not all the same).