ERTH 430: Fluid Dynamics in Earth Systems

Steps Toward Solving the Scaled Governing Equations for Waves in our Wave Tank

Dr. Dave Dempsey
Earth & Clim. Sci.
SFSU, Fall 2017

Based on our observations of periodically forced waves in the wave tank in our classroom, we can conclude the following:

1. **In the direction across the tank** (that is, from the front of the tank to the back), there is very little variation in the structure of waves except perhaps very close to the walls of the tank, where friction is the most important. We set up a coordinate system with the *x*-axis oriented along the tank (with *x* increasing from inside the tank toward the wave generator), the *z*-axis oriented vertically upward, and the *y*-axis across the tank (increasing from inside the room toward the north wall of the classroom). In this coordinate system, based on our observations, *we neglect all derivatives with respect to y in the governing equations*.

2. **Water density** varies with temperature, pressure, and composition (notably the concentration of dissolved substances, such as salts in the ocean). The water in our wave tank comes via one of the water faucets in the classroom from a storage tank somewhere in the building. We aren't supposed to drink it, but that's just a precaution—the source is the same as it is for water from drinking faucets in the hallway, which is potable (drinkable). That means it is relatively clean and has only very small amounts of stuff dissolved in it. Hence, variations in dissolved stuff is unlikely to cause water density in our tank to vary much.

The temperatures that we measured in the tank varied from place to place by around 0.5°C but hardly at all over the time it took for a wave to pass fixed location. That was a couple of hours after fresh water was added to water that had been in the tank for many weeks, which could have produced some of the variability in temperature. There could also have been problems with the measurements themselves, since we were inexperienced with the measurement technology. Once the water has been sitting in the classroom for a while, temperature variations will typically become smaller as the water and room temperatures equalize through conduction and radiative absorption and emission. Based on the seawater equation of state calculator at http://fermi.jhuapl.edu/denscalc.html, variations of 0.5°C from place to place in the tank produce variations in water density of less than 0.1 kg/m^3 (0.01% of the typical density of pure water at room temperature and sea level, which is about 1000 kg/m^3).

Scaling the Governing Equations in Tendency Form for Waves in a Wave Tank

The same calculator shows that density varies over the depth of the tank due to variations in pressure by less than 0.001 kg/m³ (0.0001%).

These are quite small variations. With further quantitative analysis (not pursued here), we can conclude that *density of water in the tank is virtually uniform (the density gradient is negligible) and steady (the density tendency is negligible)*. From this we can conclude (by invoking the relation between material and partial derivatives) that *water parcels in the tank essentially conserve their density (that is, they are essentially incompressible)*.

3. The **component of (water) velocity in the direction across the tank (*v*)** is zero. (This is consistent with our conclusion in item (1) above.)

4. The **vertical component of (water) velocity, *w*, at the bottom of the tank** is zero. (This makes sense: if *w* were non-zero, water would be moving through the bottom of the tank, but we don't see leaks.)

5. The **horizontal component of (water) velocity in the direction along the tank, *u*,** probably doesn't vary much with respect to vertical position (z) except near the bottom of the tank. Even if we accept that u varies with z, if that variation is roughly linear with z until near the bottom of the tank, then the curvature of the $u(z)$ profile is negligible (except near the bottom).

These observations and conclusions allow us to neglect some of the terms in the governing equations as they apply to the waves. The equations become:

<u>Velocity tendency equations</u> applied to waves in our wave tank:

(1)(a) $$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

(1)(b) $$\frac{\partial v}{\partial t} = 0$$

(1)(c) $$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

Scaling the Governing Equations in Tendency Form for Waves in a Wave Tank

Temperature tendency equation applied to waves in our wave tank:

(2) $$\frac{\partial T}{\partial t} = 0$$

Density tendency equation (continuity equation in tendency form) applied to waves in our wave tank:

The density is essentially steady, so:

$$\frac{\partial \rho}{\partial t} = 0$$

and the density in the tank is virtually uniform, so

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial z} = 0$$

The combination of the steadiness and uniformity of density implies that the material derivative of water density is negligible (invoking the relation between material and partial derivatives); in other words, water parcels are essentially incompressible as far as waves in our wave tank are concerned. This implies that:

(3) $$0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

Equation of State

A complex, empirical relationship among pressure, temperature, density, and composition of the liquid (careful lab observations fitted to a set of polynomial curves)

Now we scale the remaining terms in the equations above (that is, we estimate the size of the terms to the nearest order of magnitude). To do this, we draw on the Eulerian model of $u(x,z,t)$ and $w(x,z,t)$ in the wave pattern described in Lab #2, Part II (repeated below), and our observations of the characteristics of the waves (wave length, period, and half-amplitude) and the motions of water parcels in them (the horizontal and vertical half-amplitudes of their elliptical orbits).

Scaling the Governing Equations in Tendency Form for Waves in a Wave Tank

Eulerian model of the fields of u and w in our wave tank:

(4)(a) $u(x,z,t) = -A_x(z)(2\pi/T)\cos\left(\frac{2\pi}{L}(x+ct)\right)$

(4)(b) $w(x,z,t) = -A_z(z)(2\pi/T)\sin\left(\frac{2\pi}{L}(x+ct)\right)$

Substituting (4)(a) and (b) into Eqs. (1)(a) and (c) gives us:

(5)(a) $A_x(z)(2\pi/T)^2\sin(\theta(x,t)) =$
$(A_x(z))^2(2\pi/T)^2(2\pi/L)\sin(\theta(x,t))\cos(\theta(x,t)) +$
$A_z(z)\frac{\partial A_x(z)}{\partial z}(2\pi/T)^2\sin(\theta(x,t))\cos(\theta(x,t)) - \frac{1}{\rho}\frac{\partial p}{\partial x}$

5(c) $-A_z(z)(2\pi/T)^2\cos(\theta(x,t)) =$
$-A_x(z)A_z(z)(2\pi/T)^2(2\pi/L)\cos(\theta(x,t))\sin(\theta(x,t)) -$
$A_z(z)\frac{\partial A_z(z)}{\partial z}(2\pi/T)^2\sin^2(\theta(x,t)) - \frac{1}{\rho}\frac{\partial p}{\partial z} - g +$
$\frac{\mu}{\rho}A_z(z)(2\pi/T)(2\pi/L)^2\sin(\theta(x,t))$

where $\theta(x,t) \equiv \frac{2\pi}{L}(x+ct)$ and $c = L/T$.

Estimate the maximum magnitude of each term over all times (t) and locations (x,z) in the tank. The sine and cosine functions have maximum values of 1, but because they are out of phase by 90°, their product has a maximum magnitude of ½. Factors that vary with z typically are largest at the surface of the water, $z = h(x,t)$, so we'll evaluate them there (or at the mean depth of the water, $\bar{h}$). The maximum magnitudes are therefore as follows:

Max magnitudes of terms in the *u*-component velocity equation				
Term	$\partial u/\partial t$	$-u\partial u/\partial x$	$-w\partial u/\partial z$	$-(1/\rho)\partial p/\partial x$
Max	$A_x(\bar{h})(2\pi/T)^2$	$A_x(\bar{h})^2(2\pi/T)^2$ $\times(2\pi/L)/2$	$A_z(\bar{h})\frac{\partial A_x(z)}{\partial z}$ $\times(2\pi/T)^2/2$	?
Value				

Scaling the Governing Equations in Tendency Form for Waves in a Wave Tank

Max magnitude of terms in the *w*-component velocity equation						
Term	$\partial w/\partial t$	$-u\partial w/\partial x$	$-w\partial w/\partial z$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	g	$\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}\right)$
Max	$A_z(\bar{h}) \times(2\pi/T)^2$	$A_x(\bar{h})A_z(\bar{h}) \times(2\pi/T)^2 \times(2\pi/L)/2$	$A_z(\bar{h})\frac{\partial A_z(z)}{\partial z} \times(2\pi/T)^2/2$	?	g	$\frac{\mu}{\rho}A_z(\bar{h}) \times(2\pi/T) \times(2\pi/L)^2$
Value						

Max magnitude of terms in the (incompressible) continuity equation		
Term	$\partial u/\partial x$	$\partial w/\partial z$
Max	$A_x(\bar{h})\times(2\pi/T)\times(2\pi/L)$	$\frac{\partial A_z(z)}{\partial z}\times(2\pi/T) \approx \frac{A_z(\bar{h})}{\bar{h}}\times(2\pi/T)$
Value		

The scaled equations, with only the largest terms retained, should become:

(6)(a) $$\frac{\partial u}{\partial t} = \left[-u\frac{\partial u}{\partial x}\right] - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

(6)(c) $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

(7) $$0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

(The term in brackets in Eq. (6)(a) we could or should drop because it is smaller than the other terms, but keeping it retains some interesting behavior of the waves.) These three equations constrain three unknowns: u, w, and p.

Scaling the Governing Equations in Tendency Form for Waves in a Wave Tank

Our next steps toward solving the equations are as follows:

A. Integrate Eq. (6)(c) vertically between an arbitrary level, z (measure positively upward from the bottom of the tank) and the water surface at height $h(x,t)$ and solve for $p(x,t)$. (Ignore the variation of u with z for this purpose. This isn't quite correct, but it isn't bad, either.)

B. Integrate Eq. (7) vertically from the bottom of the tank to the water surface and solve for $w(h,t)$, the vertical velocity of water at the surface. (Note: if we ignore the variation of u with z here, too, then how does $\partial u/\partial x$ vary with z?) Then replace $w(h,t)$ with its definition (which involves a material derivative), and finally invoke the relation between material and partial derivatives and solve for the local derivative.