In-Class Exercise:
Mass Conservation:
Continuity Equation, Part II

(for classes starting Friday, Nov. 17)

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In Part I of this exercise, we derived a relation between the material derivative of a parcel's density, ρ, and the divergence of the velocity field (expressed in rectangular coordinates):

$$\frac{D\rho}{Dt} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

where u, v, and w are, respectively, the components of fluid velocity in the x-, y-, and z-directions in rectangular coordinates. Each of the three contributions to the velocity divergence above represents the gradient of a velocity component along the corresponding coordinate axis, which (in finite difference form) you can think of as telling us something about the difference between that velocity component between one side of a very small parcel and the opposite side. This in turn tells us how rapidly the parcel's sides are getting closer together or farther apart in that coordinate direction, and the sum of these terms in all three coordinate direction tells us something about how rapidly the parcel's volume (and hence density) is changing.

As important as conservation laws such as this one are, for practical purposes we usually prefer to work with tendency equations, which tell us something about how fast physical properties of a fluid change at a fixed location and by what mechanisms. Hence, we want a tendency equation for density.

One way to get a density tendency equation is to invoke the relation between the material derivative and partial derivatives (which is a mathematical relation, not a conservation law, but relates the material and local derivatives, which is what we need):

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\\ &= \frac{\partial\rho}{\partial t} + V\frac{\partial\rho}{\partial s} + w\frac{\partial\rho}{\partial z}\end{aligned}$$

(in rectangular and in natural coordinates, respectively).

Substituting this relation into the conservation law for density and solving for the density tendency gives us a version of the relation we want:

$$\frac{\partial \rho}{\partial t} = \left(-u\frac{\partial \rho}{\partial x} - v\frac{\partial \rho}{\partial y} - w\frac{\partial \rho}{\partial z} \right) - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

where the first term on the right-hand side (in parentheses) is the advection of density (expressed in rectangular coordinates), describing the transport by the wind of density from nearby to replace the density of departing fluid parcels; and the second term in parentheses (also expressed in rectangular coordinates) is the velocity divergence term, which describes changes in density experienced by parcels as they pass through the fixed location in question.

There is a complementary, useful way to derive this equation, which also produces a different way of writing (and interpreting) the right-hand side. It starts with **Figure 2**, which shows a fixed volume in space (not a fluid parcel) with fluid flowing through it. The idea is to relate $\frac{\partial \rho}{\partial t}$ (the rate at which the density of fluid changes at a fixed location) to the rate at which the amount of mass in a fixed, finite volume changes as a result of the net flow of fluid into or out of the volume through its faces, and take the limit as the size of the volume becomes infinitesimally small.

By picking a volume in the shape of a rectangular parallelepiped, rectangular coordinates become the natural choice for a coordinate system to describe locations and velocities.

Figure 2 shows the x-component of fluid velocity (u) at the left-hand and right-hand faces of the volume. (Note that they don't have to be positive as shown in the figure—either or both could be flowing the other direction, but regardless, they would be represented symbolically by u, where u could represent a positive or negative value.) There might be y- or z-components of velocity at each face, too, but those don't contribute to the flow of fluid into or out of the volume because they are parallel to the left and right-hand faces.

The density of the fluid flowing into or out of each face is represented by ρ at the corresponding face. (The density might be different on the different faces.)

The coordinates of the various faces are indicated in the diagram.

Over a short time, Δt, a certain mass of fluid flows into or out of each face (shown by the narrow "slabs" of fluid along each face in **Figure 2**), and the difference between (or sum of, depending on the direction of flow on each face) the masses of the two slabs determines the net change in mass in the volume over that time period due to any differences between the (face-normal) velocity components and/or densities on the two faces.

Your task: Relate the tendency (local derivative) of density to the other quantities in the diagram (namely the density on each face and the velocity components normal to each face). Start with the net change that occurs in the mass of fluid in the volume due to any difference in the flow of mass through the left and right faces. Then extend your results to include the net flows of mass through the front and back (that is, in the *y*-direction) and through the top and bottom (that is, in the *z*-direction) and sum the changes to get the total change. Then estimate the rate of change of mass in the fixed volume.

Hint: To determine the mass in a thin slab of fluid entering or leaving the fixed volume through a face, you'll need to estimate the volume of the slab. To do that, you'll need to estimate its depth. Relate that depth to other quantities given in the problem.

When you have a tendency relation for the finite volume, take the limit, both as $\Delta t \to 0$ and as the size of the volume becomes infinitesimally small.

<u>Figure 2</u>: A Rectangular Parallelepiped-Shaped Volume at a Fixed Location, with Fluid Flowing through It