ERTH 430: Fluid Dynamics in Earth Systems

In-Class Exercise: Mass Conservation: The Continuity Equation, Part I

(for classes starting Wednesday, Nov. 15)

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The Principle of Conservation of Mass can be expressed mathematically in this unexciting form:

$$Dm/Dt = 0$$

where *m* is the mass of a bit of material (e.g, a fluid parcel) and the derivative is a material (Lagrangian) derivative. This says, baldly, that the mass of a bit of matter (such as a fluid parcel) doesn't change. (This makes sense from the definition of a bit of matter such as a parcel, which is a bit of matter that maintains its identity for at least a short time, in spite of the random motions of its molecules relative to their mean motion, which is the motion of the parcel as a whole.)

This version of the Principle Conservation of Mass isn't very useful. A more useful version of the principle seems hardly a statement of conservation of mass at all:

$$D\rho/Dt = D\left(m/V\right)/Dt = \sum \text{ sources \& sinks of } \rho$$

where *V* is the volume of the parcel. This version of the principle might be more accurately called the Principle of Conservation of Density, though I don't think anyone actually calls it that.

What are the sources and sinks of density for a fluid parcel? Whatever they are, they don't change the mass of the parcel because of the definition of a parcel. However, the volume of the parcel is another story—changing the volume would change the density. How might the volume change?

There are several ways to answer this question, but the one that we usually adopt starts by recognizing that even though the mass of a parcel doesn't change, its volume can, and apply the product rule and the chain rule of calculus to $D\left(m/V\right)/Dt$:

$$\frac{D\rho}{Dt} = \frac{D(m/V)}{Dt} = \frac{Dm}{Dt}\times\left(\frac{1}{V}\right) + m\times\frac{D(1/V)}{Dt}$$

$$= 0 \qquad - m\times\left(\frac{1}{V^2}\right)\frac{DV}{Dt} = -\left(\frac{\rho}{V}\right)\frac{DV}{Dt}$$

The Mass Continuity Equation

Not surprisingly, the rate at which a parcel's density varies with respect to time is proportional to the rate at which its volume changes with respect to time. You could also say that $D\rho/Dt$ is proportional to the fractional rate of change of volume, which is $(1/V)\,DV/Dt$. (As you can see, $D\rho/Dt$ is also proportional to ρ itself.)

(1) Note the minus sign attached to $(\rho/V)DV/Dt$, which appears when we apply the chain rule to evaluate $D(1/V)/Dt =$ $D(V^{-1})/Dt = -1 \times V^{-2} \times DV/Dt = -(1/V^2)\,DV/Dt$. *Physically speaking, why must the minus sign be there?*

We are going to develop a relation between $(1/V)DV/Dt$ and (an)other familiar quantit(y)ies, and the result (multiplied by $-\rho$) will be the "source" or "sink" of our parcel's density. Take a deep breath!

Consider the rectangular parallelepiped parcel of fluid shown in the accompanying **Figure 1**. There's nothing physically special about such a shape for our purposes—it just makes it easier for us to think about certain questions and to express things mathematically. The same reasoning that we apply to a rectangular parallelepiped would apply to other shapes, but the problem might be harder to visualize and to express mathematically.

To describe the positions of the six faces of the rectangular parallelepiped, we introduce a rectangular (or Cartesian) coordinate system (with axes labeled *x*, *y*, and *z*), which is perfectly suited to a shape such as this one. We refer to the six faces as the left, right, front, back, bottom, and top faces, based on their position viewed from our perspective looking at **Figure 1**. Their respective coordinates are:

- the position of the left-hand face of the parcel $\equiv \left(x_p(t)\right)_1$
- the position of the right-hand face of the parcel $\equiv \left(x_p(t)\right)_2$
- the position of the front face of parcel $\equiv \left(y_p(t)\right)_1$
- the position of the back face of the parcel $\equiv \left(y_p(t)\right)_2$
- the elevation or altitude of the bottom of the parcel $\equiv \left(z_p(t)\right)_1$
- the elevation or altitude of the top of the parcel $\equiv \left(z_p(t)\right)_2$

The Mass Continuity Equation

(2) (a) Write the volume (*V*) of the parcel at time *t* in terms of the coordinates of the six faces at time *t* and substitute for *V* in $-(\rho/V)DV/Dt$.

(b) Apply the product rule in your results from **(2)(a)** (since the volume will be the product of the three coordinate differences, one in each coordinate direction), and perform whatever cancellations that you can in each of the three terms in the expression you get for $-(\rho/V)DV/Dt$.

It's important to note that *in each opposite pair of faces, the two faces won't necessarily be moving at the same speed, or even in the same direction*. As a result, the parcel won't necessarily maintain its shape as it moves. For simplicity, though, we'll assume opposite faces at least remain parallel to each—that is, faces move only in directions normal to themselves. Its shape might still change, but it will remain a rectangular parallelepiped. (Relaxing this assumption doesn't change the results.)

(3) (a) Rewrite your results from **(2)(b)** using common shorthand notations for the six (material) time derivatives that should appear.

(b) Do you see any potential new derivatives (in finite difference form) appearing in your results from **(3)(a)**? What are they? [*Hint: Look for potential spatial derivatives. There might be ways to rewrite your results further to help make them clearer*.]

Take the limit as the size of the parcel becomes infinitesimally small (so that each opposite pair of faces comes closer and closer together, still at time *t*). Rewrite your expression for $-(\rho/V)DV/Dt$ in terms of (standard) shorthands for the new derivatives that should have appeared.

If the instructions above worked, then you should have derived a new way of writing the source or sink of density for a parcel that involves a quantity called the ***velocity divergence***. The interpretation is really simple: *If opposite faces of a parcel move at different speeds and/or different directions, they will move closer together or further apart, and the net effect of each pair of opposite faces doing this might change the volume of the parcel and hence the density of the parcel.*

Figure 1: A Rectangular Parallelepided-Shaped Parcel of Fluid Material in Space (e.g., Air, Water, Magma, or Lava)